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Finite groups all of whose abelian subgroups are QTI-subgroups ☆

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ABSTRACT

A subgroup H of a group G is called a QTI-subgroup if $C_G(x) \leqslant N_G(H)$ for any $1 \neq x \in H$, and a group is called AQTI-group if all of its abelian subgroups are QTI-subgroups. In this paper, we obtain a classification of the finite AQTI-groups.

1. Introduction

Throughout the following, G always denotes a finite group. A subgroup H of G is called a TI-subgroup if $H \cap H^x = H$ or 1 for any $x \in G$. A topic of some interest is to investigate the finite groups in which certain subgroups are assumed to be TI-subgroups. In [7], Walls classified the finite groups all of whose subgroups are TI-subgroups. In [6] and [2], Guo, Li and Flavell classified the finite groups whose abelian subgroups are TI-subgroups. The aim of this paper is to study the finite AQTI-groups, that is, all of whose abelian subgroups are QTI-subgroups. We obtain a classification of the AQTI-groups in Theorem 2.3 (nilpotent case) and Theorem 3.3 (nonnilpotent case).

Definition 1.1. A subgroup H of G is called a QTI-subgroup if $C_G(x) \leqslant N_G(H)$ for any $1 \neq x \in H$.

Clearly a TI-subgroup is a QTI-subgroup. However, the converse is not true.

Example 1.2. Let V be an elementary abelian 3-group of order 3^5 and H be a subgroup of $GL(5,3)$ of order 11^2. Let $G = HV$, where H acts on V in a natural way.

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Since 11 does not divide $3^a - 1$ for any $a < 5$, the actions of H and its nonidentity subgroups on V are irreducible and fixed-point-free. It follows that $N_G(W) = V$ for any proper subgroup W of V and that $C_G(w) = V$ for any $1 \neq w \in W$, and therefore W is a QTI-subgroup of G. In fact, it is not difficult to see that all abelian subgroups of G are QTI-subgroups, and therefore G is an AQTI-group.

Let W_0 be a subgroup of V of order 3^4. Since $|W_0 \cap W_0^x| = 3^3$ for any $1 \neq x \in H$, W_0 is not a TI-subgroup.

Lemma 1.3. *Let G be an AQTI-group. Then the following statements hold.*

(1) *Any subgroup of G is again an AQTI-group.*
(2) *For any abelian subgroup H of G, if $H \cap Z(G) > 1$, then H is normal in G.*
(3) *For any $1 \neq x \in G$, $C_G(x)$ is nilpotent.*

Proof. (1) and (2) are clear.

(3) For any cyclic subgroup $A/\langle x\rangle$ of $C_G(x)/\langle x\rangle$, A is an abelian subgroup of an AQTI-group $C_G(x)$, and so A is normal in $C_G(x)$ (see (2)). It follows that all cyclic subgroups (and so all subgroups) of $C_G(x)/\langle x\rangle$ are normal in $C_G(x)/\langle x\rangle$. Then $C_G(x)/\langle x\rangle$ is nilpotent, and so $C_G(x)$ is nilpotent. □

Recall that a *CN*-group is a group in which the centralizer of any nonidentity element is nilpotent. Now the above lemma implies that an AQTI-group is a *CN*-group.

For any finite group G, we define its prime graph $\Gamma(G)$ (see [8]) as follows: The vertex set is $\pi(G)$, and two vertices p, q are joined by an edge if G has an element of order pq. If σ is the vertex set of a connected component of $\Gamma(G)$, then σ is called a prime component of G.

Lemma 1.4. *(See [2, Theorem 2.2].) Let G be a CN-group and σ a prime component of G. Then G possesses a nilpotent Hall σ-subgroup H, and any σ-subgroup is contained in some G-conjugate of H, furthermore H is a TI-subgroup if in addition $|\sigma| \geqslant 2$.*

In particular, if G is a nonnilpotent AQTI-group, then $\Gamma(G)$ is disconnected.

2. Nilpotent case

Recall that a Hamiltonian group is a nonabelian group in which all subgroups are normal. It is known that a Hamiltonian group is a direct product of Q_8, an elementary abelian 2-group and an abelian group of odd order. For a p-group G, we put $\mho_1(G) = \langle x^p \mid x \in G\rangle$.

Theorem 2.1. *For a finite p-group G, the following statements are equivalent.*

(1) *All subgroups of G are TI-subgroups.*
(2) *All abelian subgroups of G are TI-subgroups.*
(3) *All abelian subgroups of G are QTI-subgroups, i.e., G is an AQTI-group.*
(4) *G is one of the following p-groups*:
   (4.1) *G is an abelian p-group.*
   (4.2) *G is a Hamiltonian 2-group, that is a product of Q_8 and an elementary abelian 2-group.*
   (4.3) *G is the central product of Q_8 and D_8.*
   (4.4) *$G/Z(G)$ is of order p^2, $Z(G)$ is cyclic and $G' \cong Z_p$ is the only minimal normal subgroup of G.*

Proof. Suppose that G is of type (4.1) or (4.2). Then all subgroups of G are normal, and so they are TI-subgroups. Suppose that G is of type (4.3) and let H be a nonnormal subgroup of G. Then $H \cap Z(G) = 1$ and so H is elementary abelian. Note that the largest elementary abelian subgroup of G is of type $Z_2 \times Z_2$ (see [3, Chapter 3, Theorem 13.8]), it follows that $H \cong Z_2$, and so H is a TI-subgroup of G. Suppose that G is of type (4.4) and let H be a nonnormal subgroup of G. Then $H \cap Z(G) = 1$ and so H is of order p, thus H is a TI-subgroup.

Now we need only to show (3) implies (4).

Suppose that all abelian subgroups of G are normal. Then all subgroups of G are normal, and so G is of type (4.1) or type (4.2). In what follows, we assume that G has an abelian but not normal subgroup. Observe first that for any nontrivial abelian subgroup A of G, A is normal in G iff $A \cap Z(G) > 1$ (see Lemma 1.3(2)).

Step 1. $Z(G)$ is cyclic.

Suppose that $Z(G)$ is not cyclic and let A be any abelian subgroup of G. If $A \cap Z(G) > 1$, then A is normal in G. If $A \cap Z(G) = 1$, then AU, AV are normal in G where $U, V \cong Z_p$ are distinct subgroups of $Z(G)$, and so $A = AU \cap AV$ is normal. This implies that all abelian subgroups are normal, which contradicts our assumption.

Step 2. Let Z be the unique minimal normal subgroup of G. Then G/Z is abelian, and $Z = G'$.

Let A/Z be any cyclic subgroup of G/Z. Then A is normal in G because A is abelian with $A \cap Z(G) \geqslant Z$. It follows that all subgroups of G/Z are normal.

Suppose G/Z is nonabelian. Then G is a Hamiltonian 2-group, and so $G/Z \cong Q_8 \times Z_2 \times \cdots \times Z_2$. Let $T/Z \cong Q_8$. Clearly T is normal in G and so T' is normal in G. Since Z is the unique minimal normal subgroup of G, $T' \geqslant Z$, and this implies that $|T/T'| = 4$. Now applying [3, Chapter 3, Theorem 11.9], we conclude that $Z(T) = Z$. But it is known that $G/Z(G)$ cannot be isomorphic to a generalized quaternion group (see [3, page 94, Exercise 58]). Hence we get a contradiction. Thus G/Z is abelian, and so $Z = G'$.

Step 3. Final proof.

Since $G' = Z$ is the unique minimal normal subgroup of G, it follows by [5, Lemma 12.3] that $G/Z(G)$ is elementary abelian and that all nonlinear irreducible complex characters of G have degree $\sqrt{|G/Z(G)|}$.

Since G has an abelian but not normal subgroup A and $A \cap Z(G) = 1$, we can find an element t such that $\langle t \rangle \cap Z(G) = 1$. Then $H =: C_G(t) < G$. Observe that $|H| = |C_G(t)| \geqslant |G/G'| = |G|/p$. It follows that H is a maximal subgroup of G. Clearly H is an AQTI-group with noncyclic $Z(H)$. Using the same argument as in step 1, we conclude that all abelian subgroups of H are normal. This implies that either H is abelian or $H = Q_8 \times Z_2 \times \cdots \times Z_2$.

Suppose that H is abelian. Since $|G : H| = p$, all nonlinear irreducible complex characters of G have degree p, and this implies that $|G/Z(G)| = p^2$, thus G is of type (4.4).

Suppose that $H = Q_8 \times Z_2 \times \cdots \times Z_2$. Then G possesses an abelian subgroup of index 4. It follows that all nonlinear irreducible complex characters of G have degree 2 or 4. Thus either $|G/Z(G)| = 4$ and then G is of type (4.4), or $|G/Z(G)| = 2^4$. Let us investigate the case when $|G/Z(G)| = 2^4$. Since $Z(H)$ is elementary abelian, $Z(G) \leqslant Z(H)$ is elementary abelian, and so $Z(G) = Z$. Now $Z(G) = G' = \Phi(G)$, and thus G is an extraspecial 2-group of order 2^5. Assume that G is a central product of D_8 and D_8. Then G has a normal abelian subgroup $L \cong Z_2 \times Z_2 \times Z_2$ (see [3, Chapter 3, Theorem 13.8]). Let $t' \in L - Z$. Arguing as in the above, we conclude that $C_G(t')$ is a Hamiltonian 2-group of order 16, and so $L \leqslant C_G(t') \cong Q_8 \times Z_2$ which is clearly impossible. Now [3, Chapter 3, Theorem 13.8] implies that G is a central product of D_8 and Q_8, and hence G is of type (4.3) $\square$

Lemma 2.2. *Let G be a finite group. Then G is an AQTI-subgroup iff G satisfies the following conditions*:

(1) *G is a CN-group.*
(2) *Let σ be any prime component of G and let M be a Hall σ-subgroup of G. Then either M is one of the p-groups listed in Theorem* 2.1, *or M is abelian, or M is a Hamiltonian group.*

Proof. Suppose that G is an AQTI-group. By Lemma 1.4, G is a *CN*-group, and G possesses a nilpotent Hall σ-subgroup M for any prime component σ of G. Clearly M is again an AQTI-subgroup, and we need to show that if $|\sigma| \geqslant 2$ then all subgroups of M are normal in M. Assume this is not true. Write $M = P \times Q$, where Q is a nontrivial p'-group, and $P \in Syl_p(M)$ has an abelian but not normal subgroup P_1. Let $1 \neq x \in Z(Q) \leqslant P_1 \times Q$. As $P_1 \times Z(Q)$ is a QTI-subgroup of M, $M = C_M(x) \leqslant N_M(P_1 \times Z(Q)) = N_P(P_1) \times Q$, and this implies that P_1 is normal in P, a contradiction.

Suppose conversely that G satisfies the conditions of Lemma 2.2. Let H be an abelian subgroup of G and $1 \neq x \in H$. Let p be a prime divisor of $|H|$ and let σ be a prime component containing p of G. By Lemma 1.4 we may assume $C_G(x) \leqslant M$. If $|\sigma| \geqslant 2$, then M is a Hamiltonian group or an abelian group, thus H is normal in M, and so $C_G(x) = C_M(x) \leqslant M = N_M(H) \leqslant N_G(H)$. If $|\sigma| = 1$, then M is an AQTI-group of prime power order, so $C_G(x) = C_M(x) \leqslant N_M(H) \leqslant N_G(H)$. Thus H is a QTI-subgroup of G, and therefore G is an AQTI-group. □

Applying Theorem 2.1 and Lemma 2.2, we obtain the following result.

Theorem 2.3. *Let G be a nilpotent group. Then G is an AQTI-group if and only if one of the following holds.*

(1) *G is abelian.*
(2) *G is a Hamiltonian group.*
(3) *G is of type* (4.3) *or* (4.4) *in Theorem* 2.1.

3. Nonnilpotent case

If $G = HN$ is a Frobenius group with a kernel N and a complement H, then we say that H acts frobeniusly on N. In this case, we know that N is nilpotent and any Sylow subgroup of H is either a cyclic group or a generalized quaternion group, and that $\pi(H)$, $\pi(N)$ are the prime components of G (see [8]).

If there are $M, N \lhd G$ such that G/N is a Frobenius group with M/N as its kernel and M is a Frobenius group with N as its kernel, then G is called a 2-Frobenius group, and such a 2-Frobenius group is denoted by $Frob_2(G, M, N)$. In this case, we know that G is solvable, and that $\pi(M/N)$ and $\pi(G/M) \cup \pi(N)$ are the prime components of G (see [8]).

Lemma 3.1. *Let $G = HN$ be a Frobenius group with a complement H and a kernel N. If G is an AQTI-group, then the following statements hold.*

(1) *H is either a cyclic group or a product of Q_8 with a cyclic group of odd order.*
(2) *N is either an abelian group or of type* (4.4) *listed in Theorem* 2.1.

Proof. Since G is a Frobenius group, $\Gamma(G)$ has just two connected components $\pi(H)$ and $\pi(N)$.

(1) If H is nonnilpotent, then Lemma 1.4 implies that $\Gamma(H)$ is disconnected, and then $\Gamma(G)$ has at least three connected components, a contradiction. Thus H is nilpotent. If $P \in Syl_P(H)$ is not cyclic, then P is a generalized quaternion group, and then $P \cong Q_8$ by Theorem 2.1. The result follows.

(2) Since N is the Frobenius kernel, N is nilpotent. Assume that N is nonabelian and let P be a nonabelian Sylow p-subgroup of N. Then P is one of the three types listed in Theorem 2.1. Assume that $P \cong Q_8 \times Z_2 \times \cdots \times Z_2$. Then $\mho_1(P)$ is a normal subgroup of G of order 2, which is clearly impossible. Assume that P is the central product of Q_8 and D_8. Then $Z(P)$ lies in $Z(G)$, a contradiction. Thus P is of type (4.4) in Theorem 2.1, and then $N = P$ by Theorem 2.3. □

Lemma 3.2. *Let $G = Frob_2(G, H, K)$. If G is an AQTI-subgroup, then G is isomorphic to the symmetric group S_4.*

Proof. Note that G is solvable with just two prime components $\pi_1 = \pi(H/K)$ and $\pi_2 = \pi(G) - \pi_1$, and that G has a nilpotent Hall π_2-subgroup W (see Lemma 1.4). Clearly K is the Fitting subgroup of G, thus $C_W(K) \leqslant C_G(K) \leqslant K$, and so $W > K > Z(W)$.

Let $p \in \pi(G/H)$ and P be a Sylow p-subgroup of W. Since $K > Z(W) \geqslant Z(P)$, $P \cap K \geqslant Z(P)$ is nontrivial. Let $G_1 > P$ be a $\pi_1 \cup \{p\}$-Hall subgroup of G. It follows that $G_1 = Frob_2(G_1, H \cap G_1, P \cap K)$. Assume that $G_1 < G$. Then induction yields that $G_1 \cong S_4$, thus $P \in Syl_2(S_4)$ is isomorphic to D_8, and then $W = P$ by Theorem 2.3, so $G \cong S_4$ as wanted. Hence we may assume that $G_1 = G$. In particular, $\pi_2 = \{p\}$. Then W is one of the nonabelian p groups listed in Theorem 2.1.

Case 1. Assume that $W \cong Q_8 \times Z_2 \times \cdots \times Z_2$. As $W > K > Z(W)$, K is a product of Z_4 and an elementary abelian 2-group. It follows that $\mho_1(K) \lhd G$ with $|\mho_1(K)| = 2$, a contradiction.

Case 2. Assume that W is the central product of Q_8 and D_8. As $W > K > Z(W)$, $|K| \in \{4, 8, 16\}$.

If K is abelian, then $K \in \{Z_4 \times Z_2, Z_4, Z_2 \times Z_2\}$ (see [3, Chapter 3, Theorem 13.8]). Now $K/\Phi(K) = Z_2$ or $Z_2 \times Z_2$, it follows that $G/K \leqslant Aut(K/\Phi(K)) \leqslant S_3$, then $|P| \leqslant 16$, a contradiction.

If K is nonabelian and of order 16, then $K \cong Q_8 \times Z_2$ or $|K/Z(K)| = 4$ with $Z(K) \cong Z_4$. For the first case, let $Z = \mho_1(K)$; and for the second case, let $Z = \mho_1(Z(K))$. Then Z is normal in G with $|Z| = 2$, a contradiction.

If K is nonabelian and of order 8, then $K \cong Q_8$ or D_8, and then $G/K \leqslant Aut(K/\Phi(K)) = Aut(Z_2 \times Z_2) = S_3$, thus $|P| = 16$, a contradiction.

Case 3. Assume that $W/Z(W) \cong Z_p \times Z_p$ and $Z(W)$ is cyclic. Then K is abelian with $|W : K| = |K : Z(W)| = p$. Note that $G = N_G(U)H = N_G(U)K$ by Frattini argument, where U is a Hall π_1-subgroup of G. Clearly $N_G(U) \cap K = N_K(U) = 1$, and so $N_G(U) \cong G/K$ is a Frobenius group with a complement of order p.

Suppose K is not elementary abelian. Then $\mho_1(K)$ is a nontrivial cyclic normal subgroup of G. Let us consider $G_1 = N_G(U)\mho_1(K)$. We see that $\mho_1(K) = Fit(G_1)$, and $N_G(U) \leqslant Aut(\mho_1(K))$ is abelian, a contradiction.

Hence K is elementary abelian, and in particular $Z(W) \cong Z_p$. Now $N_G(U) \leqslant Aut(K) = Aut(Z_p \times Z_p) = GL(2, p)$. Note that if $p > 2$, then it is easy to check that $GL(2, p)$ has no subgroup which is a Frobenius group with a complement of order p. This implies that $K \cong Z_2 \times Z_2$, and hence $N_G(U) \cong S_3$, and $G \cong S_4$. □

Theorem 3.3. *Let G be a nonnilpotent group. Then G is an AQTI-subgroup iff G is one of the following groups.*

(1) *$G = HN$ is a Frobenius group with a complement H and a kernel N, where N is abelian, and H is either a cyclic group or a product of Q_8 with a cyclic group of odd order.*
(2) *$G = HN$ is a Frobenius group with a complement H and a kernel N, where H is a cyclic subgroup of Z_{p-1} and N is a p-group of the type* (4.4) *in Theorem* 2.1.
(3) *$G \cong S_4$.*
(4) *$G \cong L_2(q)$, where $q = 5, 7, 9$.*

Proof. Suppose that $G \in \{S_4, L_2(5), L_2(7), L_2(9)\}$. Then it is easy to check that G is an AQTI-group. Suppose that G is a Frobenius group of type (1) or (2). We also conclude by Lemma 2.2 that G is an AQTI-group.

Suppose that G is a nonnilpotent AQTI-group. Then the prime graph $\Gamma(G)$ is disconnected (see Lemma 1.4).

Assume that G is solvable. It is well known that G is a Frobenius or 2-Frobenius group (see [8]), and then Lemmas 3.1 and 3.2 imply that G is of type (1), (2) or (3).

In what follows, we assume that G is a nonsolvable AQTI-group. Let $N = Sol(G)$, the maximal normal solvable subgroup of G. It follows by [8] that G has a normal series $N \lhd H \lhd G$ such that N and G/H are π-groups and H/N is a nonabelian simple group, where π is the prime component of G containing 2. Furthermore, $N = Sol(G) = Fit(G)$, $G/N \leqslant Aut(H/N)$.

Let P_1 be a nilpotent Hall π-subgroup of G (see Lemma 1.4), and $P = P_1 \cap H$.

Claim 1. *If $N > 1$, then $\pi = \{2\}$.*

Suppose that $N > 1$ and $|\pi| \geqslant 2$. By Lemma 1.4 P_1 is a TI-subgroup of G. Since $N \leqslant P_1$ is a nontrivial normal subgroup of G, P_1 is normal in G, so G is solvable, a contradiction. Thus $|\pi| = 1$ and so $\pi = \{2\}$.

Claim 2. $N = 1$.

Suppose that $N > 1$ and let E be any normal subgroup of G with $1 < E \leqslant N$. By Claim 1, $\pi = \{2\}$ and P is a 2-group.

Assume that $C_G(E)N > N$. Since H/N is simple and is a unique minimal normal subgroup of G/N, $C_G(E)N \geqslant H$. Then any odd order subgroup of H acts trivially on E, which is clearly impossible. Hence $C_G(E) \leqslant N$, and in particular $P > N > Z(P)$.

Now P is one of the 2-groups listed in Theorem 2.1. Arguing as in the proof of Lemma 3.2, we can find a normal subgroup E of G with $1 < E \leqslant N$ and $E \leqslant Z_2 \times Z_2$. It follows that $G/C_G(E) \leqslant Aut(E)$ is solvable, and so G/N is solvable because $C_G(E) \leqslant N$, a contradiction.

Claim 3. *$H \cong L_2(q)$, where $q = 5, 7, 9$.*

As $N = 1$, H is a nonabelian simple group. Since H is an AQTI-group, by Lemma 1.3(3) H is a CN-group. Note that the only simple nonabelian CN-groups are $Sz(q)$, $L_3(4)$, $L_2(9)$, and $L_2(p)$ where p is a Fermat or a Mersenne prime (see [4, Chapter XI, Remark 3.12]).

Assume that $H \cong Sz(q)$. Then $|P| = q^2$, $q = 2^{2m+1}$, where $P' \cong \Phi(P) = Z(P)$ is an elementary abelian group of order q. Checking the 2-groups listed in Theorem 2.1, we get a contradiction.

Assume that $H \cong L_3(4)$. Then $|P| = 2^6$, and $Z(P) \cong Z_2 \times Z_2$. Checking the 2-groups listed in Theorem 2.1, we get a contradiction.

Assume that $H \cong L_2(p)$, where p is a prime and $p = 2^m + 1$ or $2^m - 1$. Then P is a dihedral group of order 2^m (see [3, Chapter II, Theorem 8.27]). Checking the 2-groups listed in Theorem 2.1, we conclude that $P \cong Z_2 \times Z_2$ or D_8. Thus either $p = |P| + 1 = 5$ and then $H \cong L_2(5)$, or $p = |P| - 1 = 7$ and then $H \cong L_2(7)$.

Claim 4. *$G = H \cong L_2(q)$, where $q = 5, 7, 9$.*

It suffices to show that $G = H$. Otherwise, $H < G \leqslant Aut(H)$. We will apply [1] to get a contradiction.

Assume that $H \cong A_5$ (or $L_2(7)$). Then $G \cong S_5$ (or $PGL(2,7)$) has an element of order 6, so 2, 3 lie in the same prime component of G. However neither S_5 nor $PGL(2,7)$ has a nilpotent Hall $\{2,3\}$-subgroup, a contradiction.

Assume that $H \cong L_2(9)$. Then G contains a subgroup which is isomorphic to $L_2(9):2_1$, $L_2(9):2_2$ or $L_2(9):2_3$ (see [1]). If $L_2(9):2_1 \leqslant G$, then G has an element of order 6 but has no nilpotent Hall $\{2,3\}$-subgroup, a contradiction. If $L_2(9):2_2 \leqslant G$, then G has an element of order 10 but has no nilpotent Hall $\{2,5\}$-subgroup, a contradiction. If $L_2(9):2_3 \leqslant G$, then a Sylow 2-subgroup U of $L_2(9):2_3$ has order 16 and $|Z(U)| = 2$, and we also get a contradiction by checking the 2-groups listed in Theorem 2.1. Thus $G = H$ as desired. □

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